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CIRCLE ACTIONS ON RATIONAL HOMOLOGY MANIFOLDS AND DEFORMATIONS OF RATIONAL HOMOTOPY TYPES

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ABSTRACT. The aim of this paper is to follow up the program set in [LR85, Rau92], i.e., to show the existence of nontrivial group actions ("symmetries") on certain classes of manifolds. More specifically, given a manifold X with submanifold F, I would like to construct nontrivial actions of cyclic groups on X with F as fixed point set. Of course, this is not always possible, and a list of necessary conditions for the existence of an action of the circle group $T = S^1$ on X with fixed point set F was established in [Rau92]. In this paper, I assume that the rational homotopy types of F and X are related by a deformation in the sense of [All78] between their (Sullivan) models as graded differential algebras (cf. [Sul77, Hal83]). Under certain additional assumptions, it is then possible to construct a rational homotopy description of a T-action on the complement $X \setminus F$ that fits together with a given T-bundle action on the normal bundle of F in X. In a subsequent paper [Rau94], I plan to show how to realize this T-action on an actual manifold Y rationally homotopy equivalent to X with fixed point set F and how to "propagate" all but finitely many of the restricted cyclic group actions to X itself.

1. RATIONAL COHOMOLOGY

Given a (smooth) manifold X and a submanifold $F \subset X$ whose rational homotopy types are related in a sense to be made more precise in several assumptions throughout this paper. In this section, we would like to construct the rational *cohomology* of a candidate for the orbit space of a $T = S^1$-action on the complement $X \setminus F$ such that its (algebraic) boundary fits to the orbit space of a fibrewise T-action on the sphere bundle $S\nu$ of the normal bundle ν of F in X.

In order to formulate some assumptions relating X and F, we need to describe the rational homotopy types involved as differential graded algebras (dgas) over $\mathbf{Q}$ via their minimal models (see, e.g., [BG76, Sul77, GM81, Hal83, AP93]). The following definition is a modification of that in [Ger64] to the category of dgas:

Definition 1.1. Let $(\mathscr{A}, d)$ be a differential graded algebra over $\mathbf{Q}$, and let e denote a (formal) variable in dimension two.

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(1) The graded algebra $\mathscr{A}[e] = \mathscr{A} \otimes \mathbf{Q}[e]$ together with a differential $d_{[e]}$ is a 1-*parameter deformation* of $(\mathscr{A}, d)$ if $d_{[e]}$ projects to d under the augmentation map $\varepsilon : \mathscr{A}[e] \to \mathscr{A}$. More precisely [LR85, Satz 2.8], the differentials $d_{[e]}$ and d are related by a derivation $\tau : \mathscr{A} \to \mathscr{A}[e]$ such that
   - $d_{[e]}(e) = 0$;
   - $d_{[e]}(x) = d(x) + e \cdot \tau(x)$, $x \in \mathscr{A}$;
   - $d_{[e]} \circ \tau + \tau \circ d = 0$.

(2) The *trivial* deformation of $\mathscr{A}$ is characterized by $\tau = 0$.

(3) A 1-*parameter deformation* of a dga morphism $j : (\mathscr{A}, d) \to (\mathscr{B}, d')$ is a dga morphism $j_{[e]} : (\mathscr{A}[e], d_{[e]}) \to (\mathscr{B}[e], d'_{[e]})$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathscr{A}[e] & \xrightarrow{j_{[e]}} & \mathscr{B}[e] \\ \downarrow \varepsilon & & \downarrow \varepsilon \\ \mathscr{A} & \xrightarrow{j} & \mathscr{B}. \end{array}$$

Note that a 1-parameter deformation of $\mathscr{A}$ induces an "algebraic Gysin sequence" [LR85]

$$(1.1) \qquad \cdots \to H^{*+1}(\mathscr{A}) \xrightarrow{t=\tau^*} H^*(\mathscr{A}[e]) \xrightarrow{\cdot e} H^{*+2}(\mathscr{A}[e]) \xrightarrow{p=\varepsilon^*} H^{*+2}(\mathscr{A}) \to \cdots$$

as the cohomology long exact sequence of the short exact sequence

$$0 \to \mathscr{A}[e] \xrightarrow{\cdot e} \mathscr{A}[e] \xrightarrow{\varepsilon} \mathscr{A} \to 0.$$

Similarly, a 1-parameter deformation of a dga-morphism induces graded algebra morphisms which fit into a ladder between the corresponding Gysin sequences.

C. Allday defined in [All78] the category of $\mathbf{Z}/2$-graded augmented (Koszul-Sullivan)-differential algebras (KS2DGA). Let $R = \mathbf{Q}[e]$, and $K = \mathbf{Q}(e)$. In our context, the most important example of such a $\mathbf{Z}/2$-graded object is $(\mathscr{A}(e), d_{(e)}) = (\mathscr{A}[e] \otimes_R K, d_{[e]} \otimes_R K)$, which, as a graded algebra, is equal to $\mathscr{A} \otimes K$.

Definition 1.2. A dga morphism $j_{[e]} : (\mathscr{A}[e], d_{[e]}) \to (\mathscr{B}[e], d'_{[e]})$ is called a *local isomorphism* if and only if

$$j_{[e]} \otimes_R \mathrm{id}_K : (\mathscr{A}(e), d_{(e)}) \to (\mathscr{B}(e), d'_{(e)})$$

is a weak homotopy equivalence of *KS2DGA*s, i.e., if it induces an isomorphism in homology.

In that case, the latter morphism is in fact a homotopy equivalence in the category of KS2DGAs [All78, Proposition 2.3]

From now on, we assume that X is a smooth closed simply-connected n-dimensional manifold and that $F = \coprod_i F_i \subset X$ consists of finitely many smooth closed (simply-connected) submanifolds F_i of dimension $n_i < n$. Their rational minimal models in the sense of [Sul77, GM81, Hal83] are denoted by $(\mathscr{M}(X), d)$, resp. $(\mathscr{M}(F), d')$. Inclusion induces a dga map $j : (\mathscr{M}(X), d) \to (\mathscr{M}(F), d')$. Note that the minimal model of the space $F_T = F \times BT$ is given by $(\mathscr{M}(F)[e], d')$ with trivially extended differential. Furthermore we impose

Assumption A. *The normal bundle* $\nu = \nu(F, X) = \coprod \nu(F_i, X)$ *of* F *in* X *supports a complex structure.*

Assumption B. *There is a* 1*-parameter deformation* $(\mathscr{M}(X)[e], d_{[e]})$ *of the minimal model of* X *and a* 1*-parameter deformation* $j_{[e]} : (\mathscr{M}(X)[e], d_{[e]}) \to (\mathscr{M}(F)[e], d')$ *of the inclusion map* j *into the trivial deformation of* $\mathscr{M}(F)$, *which is a* local isomorphism.

Remark 1.3. In the assumption above, we talk about a specific manifold F. Instead, one might just require a deformation into the minimal model of a space F that has a lower cohomological dimension than X itself. It can then be shown along the lines of [Rau92] and the references there, that every component F_i is a rational Poincaré complex and that the Poincaré forms of X and F are related. Remark that the proofs in [Rau92] only use the Borel localization theorem, i.e., a situation that is guaranteed by Assumption B. In [Rau94], we shall discuss how to realize those rational Poincaré complexes by manifolds.

The map $j_{[e]}$ in Assumption B should be thought of as an algebraic simulation of the inclusion of Borel spaces $F_T \hookrightarrow X_T$, where F is the fixed point set of a T-action on X. The dgas and maps inbetween them may be realized by rational spaces and maps [BG76, Hal83], which we denote by $j_{[e]} : F_{[e]} = F_{(0)} \times BT_{(0)} \to X_{[e]}$. Also the augmentation maps $\varepsilon : (\mathscr{M}(X)[e], d_{[e]}) \to (\mathscr{M}(X), d)$ and $\varepsilon : (\mathscr{M}(F)[e], d') \to (\mathscr{M}(F), d')$ may be realized by maps $p_X : X_{(0)} \to X_{[e]}$, resp. $p_F : F_{(0)} \to F_{(0)} \times BT_{(0)}$.

Interpreting $(X_{[e]}, F_{[e]})$ as a pair of rational spaces, Assumption B shows in particular, that $H^*(X_{[e]}, F_{[e]})$ is a $\mathbf{Q}[e]$-torsion module. As in [Rau92], one may prove

Lemma 1.4. *Under Assumption* B, *the map* $j^*_{[e]} : H^n(X_{[e]}) \to H^n(X)$ *is onto in dimension* n, *if* $F \neq \emptyset$.

A choice of a complex structure on ν induces a (semifree) $T \subset \mathbf{C}^*$-action. After choice of a Hermitian metric on ν and conjugation with the associated exponential map, this action induces a semifree action on a tubular neighborhood $F \subset U \subset X$ with fixed point set F. In particular, ∂U becomes a free T-manifold T-diffeomorphic to the sphere bundle $S\nu$ with orbit space $\partial U/T$ diffeomorphic to $\mathbf{C}P\nu$. Our aim is to construct the *homology* of a (*virtual*) *orbit space* for a (free) T-action on the manifold $M = X \setminus U \simeq X \setminus F$ *with boundary* $\mathbf{C}P\nu$. The first step is:

Lemma 1.5.

$$H^*(X_{[e]}, F_{[e]}) = H^*(j_{[e]}) = \begin{cases} \mathbf{Q}, & * = n-1, \\ 0, & * \geq n. \end{cases}$$

Proof. We have to chase the diagram of pairs

$$\begin{array}{ccccccccccc}
\cdots \to H^{*+1}(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^{*+1}(F_{[e]}) & \xrightarrow{\delta} & H^{*+2}(j_{[e]}) & \to & H^{*+2}(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^{*+2}(F_{[e]}) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to H^{*-1}(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^{*-1}(F_{[e]}) & \xrightarrow{\delta} & H^{*}(j_{[e]}) & \to & H^{*}(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^{*}(F_{[e]}) & \to \cdots
\end{array}$$

The vertical arrows stand for multiplication by e, which is an isomorphism on $H^*(F_{[e]})$ for all $* \geq \max\{n_i\} - 1$ and thus for $* \geq n-3$, and on $H^*(X_{[e]})$ for $* \geq n-1$. The five lemma shows that it is an isomorphism on $H^*(j_{[e]})$ for

$* \geq n$. Furthermore, Assumption B shows that the map $j_{[e]}^{*}$ is an isomorphism for large $*$ and hence for $* \geq n$, which allows to conclude that $H^*(j_{[e]}) = 0$ for $* \geq n$.

It follows from Lemma 1.4, that $j_{[e]}^{n-1}$ is an isomorphism, too, and hence we have $H^{n-1}(j_{[e]}) \cong \operatorname{coker}(j_{[e]}^{n-2})$. On the other hand, by Lemma 1.4, multiplication with e on $H^{n-2}(X_{[e]})$ has a 1-dimensional cokernel, generated by an element $x_{[e]}$ with $p_X^*(x_{[e]}) = [X] \in H^n(X)$, the fundamental class. Hence, $H^{n-1}(j_{[e]})$ is a 1-dimensional $\mathbf{Q}$-vector space generated by $\delta(\frac{1}{e} \cdot j_{[e]}^*(x_{[e]}))$. □

The cup-product defines $\mathbf{Q}$-bilinear pairings

$$I : H^*(X_{[e]}) \otimes H^{n-1-*}(X_{[e]}, F_{[e]}) \to H^{n-1}(X_{[e]}, F_{[e]}) \cong \mathbf{Q}, \qquad 0 \leq * \leq n-1,$$

which in turn yields vector space homomorphisms

$$i_{[e]} : H^*(X_{[e]}) \to H_{n-1-*}(X_{[e]}, F_{[e]}) = \operatorname{Hom}(H^{n-1-*}(X_{[e]}, F_{[e]}); \mathbf{Q}).$$

From now on, we impose:

Assumption C. *The inclusion j of F into X induces the trivial map in rational cohomology in positive degrees.*

Proposition 1.6. *Under Assumption* C, *the maps $i_{[e]}$ above are onto.*

Proof. We use the following notation introduced in [Rau92] (in the absolute case): Let

$$R(X, F) = p^* H^*(X_{[e]}, F_{[e]}) \subset H^*(X, F),$$

and

$$I(X) = p^*(\operatorname{Tor}(H^*(X_{[e]})))$$

denote the image of the $\mathbf{Q}[e]$-*torsion* submodule of $H^*(X_{[e]})$. First, we show that the map $k^* : H^*(X, F) \to H^*(X)$ induces an isomorphism

$$K : H^*(X, F)/R(X, F) \to H^*(X)/I(X),$$

using the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to H^{*-1}(F_{[e]}) & \to & H^*(X_{[e]}, F_{[e]}) & \xrightarrow{k_{[e]}^*} & H^*(X_{[e]}) & \to & H^*(F_{[e]}) \to \cdots \\
\downarrow & & \downarrow p^* & & \downarrow p^* & & \downarrow \\
\cdots \xrightarrow{0} H^{*-1}(F) & \xrightarrow{\delta} & H^*(X, F) & \xrightarrow{k^*} & H^*(X) & \xrightarrow{0} & H^*(F) \to \cdots
\end{array}$$

The quotient map K is well-defined, since, by Assumption B, $H^*(X_{[e]}, F_{[e]})$ is a $\mathbf{Q}[e]$-torsion module. It is one-to-one, since every torsion element in $H^*(X_{[e]})$ comes from $H^*(X_{[e]}, F_{[e]})$, and since $\delta(H^{*-1}(F)) = \delta(R(F)) \subset R(X, F)$. Finally, K is onto because of Assumption C.

It is shown in [Rau92] that $R(X) = I(X)^{\perp}$ under the Poincaré duality form on $H^*(X)$. Hence, the latter factors over a nondegenrate pairing $P : H^*(X)/I(X) \to R(X)^*$, the $\mathbf{Q}$-vector space dual to $R(X)$. Furthermore, the transgression map $\tau : H^*(X, F) \to H^{*-1}(X_{[e]}, F_{[e]})$ induces an isomorphism $\tau : H^*(X, F)/R(X, F) \to \ker(\cdot e) \subset H^{*-1}(X_{[e]}, F_{[e]})$. Moreover, the duality forms P and I are linked together by the transgression τ as follows:

(1) $\tau : \mathbf{Q} \cong H^n(X, F) \to H^{n-1}(X_{[e]}, F_{[e]}) \cong \mathbf{Q}$ is an isomorphism.
(2) $\tau(y \cup p^* x) = \tau(y) \cup x$, $y \in H^*(X, F)$, $x \in H^*(X_{[e]})$ [LR85].

In the commutative diagram

$$\begin{array}{ccccc} H^*(X, F)/R(X, F) & \xrightarrow{K \cong} & H^*(X)/I(X) & \xrightarrow{P \cong} & R_{n-*}(X) \\ \tau \downarrow\cong & & & & \downarrow p_* \\ \ker(\cdot e) & \subset & H^{*-1}(X_{[e]}, F_{[e]}) & \xrightarrow{i^*_{[e]}} & H_{n-*}(X_{[e]}), \end{array}$$

the composite map $H^*(X, F)/R(X, F) \to H_*(X_{[e]})$ is monomorphic. Hence the adjoint of $i_{[e]}$ has to be injective on $\tau H*(X, F) = \ker(\cdot e)$. We conclude that $i_{[e]} : H^*(X_{[e]}) \to (\ker(\cdot e))^*$, the dual of $\ker(\cdot e)$, is onto.

To show that $i_{[e]}$ is onto $H_{n-1-*}(X_{[e]}, F_{[e]})$, pick an element

$$z \in \ker(\cdot e^i) \setminus ker(\cdot e^{i-1}) \subset H^*(X, F).$$

Since $e^{i-1} \cdot z \in \ker(\cdot e)$, there is an element $x \in H^*(X_T)$ such that $z \cup e^{i-1} \cdot p^*x = e^{i-1} \cdot z \cup p^*x \neq 0 \in H^{n-1}(X_{[e]}, F_{[e]})$. □

For $M = X \setminus F$, Alexander duality suggest the following

Definition 1.7. $H^*_{[e]}(M) = H_{n-1-*}(X_{[e]}, F_{[e]})$.

In particular, $H^*_{[e]}(M) = 0$ for $* < 0$ and $* \geq n$. The surjections $i_{[e]} : H^*(X_{[e]}) \to H^*_{[e]}(M)$ can be used to give the latter graded **Q**-vector space the structure of a graded ring with a map $(\cdot e)^*$ of degree 2 as a graded $\mathbf{Q}[e]$-quotient algebra of $H^*(X_{[e]})$.

Our next goal is to define connecting homomorphisms in a Gysin type long exact sequence

$$(1.2) \qquad \cdots H^{*-1}(M) \xrightarrow{t_M} H^{*-2}_{[e]}(M) \xrightarrow{\cdot e} H^*_{[e]}(M) \xrightarrow{p_M} H^*(M) \cdots .$$

This is quite easy using Alexander duality $A : H^*(X \setminus F) \cong H^*(X \setminus D\nu) \cong H_{n-*}(X, D\nu) \cong H_{n-*}(X, F)$. In detail, t_M, resp. p_M, are given by the compositions

$$t_M : H^{*-1}(X \setminus F) \xrightarrow{A} H_{n-*+1}(X, F) \xrightarrow{p_*} H_{n-*+1}(X_{[e]}, F_{[e]}) \xrightarrow{\cong} H^{*-2}_{[e]}(X \setminus F),$$

$$p_M : H^*_{[e]}(X \setminus F) \xrightarrow{\cong} H_{n-*-1}(X_{[e]}, F_{[e]}) \xrightarrow{t^*} H_{n-*}(X, F) \xrightarrow{A^{-1}} H^*(X \setminus F).$$

As in [LR85, Lemma 2.1.c], one may show inductively

Lemma 1.8. *The Gysin type sequence* (1.2) *is exact.*

Next, we have to simulate the inclusion of the boundary $\mathbf{C}P\nu \subset (X \setminus F)/T$ in case of a T-action by an algebraic counterpart. Geometry imposes an additional requirement: Let $F = \coprod_i F_i$, $\mathbf{C}P\nu = \coprod_i \mathbf{C}P\nu_i$ denote decomposition into connected components. Then $H^{n-2}(\mathbf{C}P\nu) \cong \bigoplus_i H^{n-2}(\mathbf{C}P\nu_i) \cong \bigoplus_i \mathbf{Q}$ by evaluation at (properly chosen) fundamental classes. Adding over the components yields a map $E : H^{n-2}(\mathbf{C}P\nu) \to \bigoplus_i \mathbf{Q} \to \mathbf{Q}$. Let $p_{[e]} : \mathbf{C}P\nu \to F_T$ denote the Borel construction applied to the inclusion $S\nu \hookrightarrow D\nu$, which is a map over F.

Assumption D. *The sequence*

$$0 \to H^{n-2}(X_{[e]}) \xrightarrow{j^*_{[e]}} H^{n-2}(F_{[e]}) \xrightarrow{E \circ p^*_{[e]}} \mathbf{Q} \to 0$$

is exact.

Remark 1.9. (1) The proof of Lemma 1.5 shows, that the cokernel of $j^*_{[e]}$ is isomorphic to $\mathbf{Q}$ under Assumption B. Hence, the assumption only specifies the image of $j^*_{[e]}$ in dimension $n-2$.

(2) If there is a semifree T-action on X with fixed point set F and with $M \simeq X \setminus F$, then the sequence in Assumption D is in fact exact, since

(a) $H^{n-2}(X_{[e]}, F_{[e]}) \cong H^{n-2}(M/T, \mathbf{C}P\nu) \overset{\cdot e}{\cong} H^n(M/T, \mathbf{C}P\nu) = 0$ (use excision and $H^{n-1}(M, S\nu) \cong H_1(M) = 0$).

(b) The following diagram commutes:

$$\begin{array}{ccccc} H^{n-2}(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^{n-2}(F_{[e]}) & \xrightarrow{\delta} & H^{n-1}(X_{[e]}, F_{[e]}) \cong \mathbf{Q} \to 0 \\ & & \downarrow p^*_{[e]} & & \downarrow\cong \\ & & H^{n-2}(\mathbf{C}P\nu) & \xrightarrow{\delta=E} & H^{n-1}(M/T, \mathbf{C}P\nu) \cong \mathbf{Q} \end{array}$$

(After proper choices of fundamental classes, the bottom map δ corresponds to the map E above.)

Corollary 1.10. *Assumption* D *implies*: $\ker(i_{[e]}) \subseteq \ker(p^*_{[e]} \circ j^*_{[e]})$, *i.e.*, $p^*_{[e]} j^*_{[e]} z = 0$ *for all* $z \in H^*(X_{[e]})$ *satisfying* $z \cup H^{n-*-1}(X_{[e]}, F_{[e]}) = 0 \in H^{n-1}(X_{[e]}, F_{[e]})$. *Hence, there is a well-defined ring homomorphism* $k_{[e]}$ *completing the square*

$$\begin{array}{ccc} H^*(X_{[e]}) & \xrightarrow{j^*_{[e]}} & H^*(F_{[e]}) \\ \downarrow i_{[e]} & & \downarrow p^*_{[e]} \\ H^*_{[e]}(X \setminus F) & \overset{k_{[e]}}{\dashrightarrow} & H^*(\mathbf{C}P\nu). \end{array}$$

Remark 1.11. If the fixed point set F is connected, the conclusion above has the following geometric interpretation: $j^*_{[e]}(z) \in H^*(F_{[e]})$ is a multiple of the total Chern class

$$C(\nu) = e^{\frac{n-m}{2}} + c_1(\nu F) \cdot e^{\frac{n-m}{2}-1} + c_2(\nu F) \cdot e^{\frac{n-m}{2}-2} + \cdots + c_{\frac{n-m}{2}}(\nu F).$$

Proof. The existence of $k_{[e]}$ above is equivalent to the first statement in the corollary. The latter is trivially true for $* \geq n-1$, since $H^*(\mathbf{C}P\nu) = 0$.

Now, suppose $z \in H^*(X_{[e]})$, $* \leq n-2$, and $z \cup H^{n-*-1}(X_{[e]}, F_{[e]}) = 0$. Let δ denote the connecting homomorphism $\delta : H^*(F_{[e]}) \to H^{*+1}(X_{[e]}, F_{[e]})$. For all $s \in H^{n-*-2}(F_{[e]})$, the condition above implies: $\delta(j^*_{[e]}(z) \cup s) = z \cup \delta s = 0$. Hence, for every such s, there is an element $u \in H^{n-2}(X_{[e]})$ with $j^*_{[e]}(u) = j^*_{[e]}(z) \cup s$.

Thus, for every $\sigma_i \in H^{n-*-2}(\mathbf{C}P\nu_i)$, Assumption D implies:

$$E(p^*_{[e]}(j^*_{[e]}(z)) \cup \sigma_i) = 0.$$

Since $p^*_{[e]}(j^*_{[e]}(z)) \cup \sigma_i$ lives in $H^{n-2}(\mathbf{C}P\nu_i)$, it has to be trivial itself. Using Poincaré duality for $\mathbf{C}P\nu_i$, we conclude that $p^*_{[e]}(j^*_{[e]}(z)) = 0$. □

We finish this section with two lemmas that show that our algebra behaves as the cohomology of an orbit space with respect to quotient maps and transgressions:

Lemma 1.12. *Let $k : S\nu \hookrightarrow X \setminus F$ denote the inclusion map. The diagram*

$$\begin{array}{ccc} H^*_{[e]}(X_{[e]}) & \xrightarrow{k_{[e]}} & H^*(\mathbf{C}P\nu) \\ \downarrow p_M^* & & \downarrow p_\nu^* \\ H^*(X \setminus F) & \xrightarrow{k^*} & H^*(S\nu) \end{array}$$

commutes. If F is rationally contractible in X, i.e., $j^* = 0$ for $* > 0$, both compositions are trivial.

Proof. The diagram of the lemma embeds into

$$\begin{tikzcd} H^*(X_{[e]}) \arrow[rrr, "j^*_{[e]}"] \arrow[dr, "i_{[e]}"] \arrow[ddd, "p_X^*"'] & & & H^*(F_{[e]}) \arrow[dl, "p^*_{[e]}"'] \arrow[ddd, "p_F^*"] \\ & H^*_{[e]}(X \setminus F) \arrow[r, "k_{[e]}"] \arrow[d, "p_M^*"] & H^*(\mathbf{C}P\nu) \arrow[d, "p_\nu^*"] & \\ & H^*(X \setminus F) \arrow[r, "k^*"] & H^*(S\nu) & \\ H^*(X) \arrow[ur, "i^*"] \arrow[rrr, "j^*"] & & & H^*(F). \arrow[ul] \end{tikzcd}$$

Since $i_{[e]}$ is surjective, it is enough to see that all the outer diagrams commute. This is true by definition apart from the left parallelogram, which rewrites as

$$\begin{array}{ccc} H^*(X_{[e]}) & \xrightarrow{i_{[e]}} & H_{n-*-1}(X_{[e]}, F_{[e]}) \\ \downarrow p_X^* & & \downarrow t^* \\ H^*(X) & \longrightarrow & H_{n-*}(X, F), \end{array}$$

where the horizontal arrows denote evaluation at the fundamental classes. From the Gysin sequence (1.2) it is easy to see (as in [LR85]) that $t : H^n(X, F) \to H^{n-1}(X_{[e]}, F_{[e]})$ is an isomorphism, and commutativity of the last diagram thus follows from:

$$t(p_X^*(y) \cup z) = y \cup t(z), \quad y \in H^*(X_{[e]}), \; z \in H^{n-*}(X, F),$$

see [LR85, p. 552]. □

Lemma 1.13. *The following diagram commutes:*

$$\begin{array}{ccc} H^*(M) & \xrightarrow{k^*} & H^*(S\nu) \\ \downarrow t & & \downarrow t \\ H^{*-1}_{[e]}(M) & \xrightarrow{k_{[e]}} & H^{*-1}(\mathbf{C}P\nu). \end{array}$$

Proof. Embed the diagram of the lemma as the center of the following diagram:

$$\begin{tikzcd}
H_{n-*}(X\,,F) \arrow[ddd, "p_*"'] & H_{n-*}(M\,,S\nu) \arrow[l, "\cong"'] \arrow[rr, "\partial"] & & H_{n-*-1}(S\nu) \arrow[ddd, "p_*"] \\
& H^*(M) \arrow[u, "\cong"'] \arrow[r, "k^*"] \arrow[d, "t"] & H^*(S\nu) \arrow[ur, "\cong"] \arrow[d, "t"] & \\
& H^{*-1}_{[e]}(M) \arrow[r, "k_{[e]}"] & H^{*-1}(\mathbf{C}P\nu) \arrow[dr, "\cong"] & \\
H_{n-*}(X_{[e]}\,,F_{[e]}) \arrow[ur, "\cong"] \arrow[d, "\partial"] & H^{*-1}(X_{[e]}) \arrow[l, "i_{[e]}"'] \arrow[u, "i_{[e]}"'] \arrow[r, "j^*_{[e]}"] & H^{*-1}(F_{[e]}) \arrow[u, "p^*_{[e]}"'] & H_{n-*-1}(\mathbf{C}P\nu) \arrow[d, "p_{[e]*}"] \\
H_{n-*-1}(F_{[e]}) \arrow[rrr, equal] & & & H_{n-*-1}(F_{[e]}).
\end{tikzcd}$$

We have to check commutativity of the outer and of the lower "rectangles"; commutativity of the smaller interior diagrams is by definition or routine.

The outer diagram commutes, since it is dual to part of the following commutative diagram:

$$\begin{tikzcd}
H^{n-*-1}(\mathbf{C}P\nu) \arrow[r, "p^*_\nu"] & H^{n-*-1}(S\nu) \arrow[dr, "\delta"] & \\
H^{n-*-1}(F_{[e]}) \arrow[u, "p^*_{[e]}"'] \arrow[r, "p^*_F"] \arrow[d, "\delta"] & H^{n-*-1}(F) \arrow[u, "p^*"'] \arrow[d, "\delta"] & H^{n-*}(M\,,S\nu). \\
H^{n-*}(X_{[e]}\,,F_{[e]}) \arrow[r, "p^*_X"] & H^{n-*}(X\,,F) \arrow[ur, "i^*"', "\cong"] &
\end{tikzcd}$$

Now to the lower part of the diagram: Let $y \in H^{*-1}(X_{[e]})$. Moving to the left, it corresponds to the linear form $(z \mapsto y \cup \delta z \in H^{n-1}(X_{[e]}\,, F_{[e]}) \cong \mathbf{Q})$ on $H^{n-*-1}(F_{[e]})$. Under the path to the right it corresponds to the map $(z \mapsto E(p^*_{[e]} j^*_{[e]} y \cup p^* z) \in \mathbf{Q})$. Both come from the map $(z \mapsto j^*_{[e]} y \cup z \in H^{n-2}(F_{[e]}))$ by composition with δ, resp. $p^*_{[e]}$. According to Assumption D, the maps δ and $E \circ p^*_{[e]} : H^{n-2}(\mathbf{C}P\nu) \to \mathbf{Q}$ agree upto a nontrivial rational factor, which can be eliminated by a change of the fundamental class $H^{n-1}(X_{[e]}\,, F_{[e]})$. $\square$

2. Rational homotopy: Perturbing spaces and maps

It was the plan of the preceding section to describe the cohomology of a potential orbit space of a T-action on $M = X \setminus F$ as a perturbation (quotient) of the cohomology of the potential Borel space $X_{[e]}$. Similarly, the cohomology of the inclusion map from $\mathbf{C}P\nu$ into it was obtained by a perturbation (quotient) of the deformation map $j^*_{[e]} : H^*(X_{[e]}) \to H^*(F_{[e]})$. In this section, we are going to refine this method to rational homotopy [Sul77, BG76, GM81, Hal83], which makes additional hypotheses necessary.

First, we show that certain conditions on the (co)-connectivity of X, resp. F, imply that the rational homotopy of X and $M = X \setminus F$ are closely related. As a consequence, it turns out, that a minimal model of a space $M_{[e]}$ may be constructed as a perturbation of $\mathscr{M}(X_{[e]})$, and similarly a rational map $\mathbf{C}P\nu \to M_{[e]}$.

We begin by presenting some necessary easy (and probably well-known) lemmas from rational homotopy theory. The first question is: Given a map

$f : A \to X$. To which extent does the rational homotopy of X together with the rational cohomology of the map f determine the rational homotopy of A, or, in other words, is the rational homotopy of A "formal, given that of X"? The reader should have in mind the case $A = X \setminus F$.

Proposition 2.1. *Let $f : A \to X$ be a map between 1-connected CW-complexes with*

$$\begin{array}{llll} H^*(X;\mathbf{Q}) = 0, & * \le i; & H^*(f;\mathbf{Q}) = 0, & * \le k; \\ H^*(A;\mathbf{Q}) = 0, & * \ge j. & & \end{array}$$

For $j \le \min\{i+k, 2k-1\}$, the rational homotopy of A is determined by that of X and by the (canonically induced) $i^* : H^*(f) \to H^*(X)$.

Proof. Let $\mathscr{A}$ denote the functor which to a simplicial complex associates its rational PL de Rham complex [Sul77, Hal83]. Regard the composition $\mathscr{M}(X) \to \mathscr{A}(X) \to \mathscr{A}(A)$ of a model map for X and the map induced by f. A model for A extending $\mathscr{M}(X)$ can be obtained from the Postnikov tower of the map f turned into a fibration $F_f \to A \to X$. This has been made explicit in the thesis of Grivel, see [Hal83]. In our case, F_f is rationally $(k-1)$-connected, and hence, up to dimension $2k-2$, the following diagram consists of isomorphisms:

$$\begin{array}{cccc} \mathscr{M}(F_f)(2k-2) \cong & \pi^*(F_f) & \cong & s^{-1}\pi^*(f) \\ & \uparrow h^* \cong & & \uparrow h^* \cong \\ & H^*(F_f) & \cong & s^{-1}H^*(f) \end{array} \qquad (* \le 2k-2).$$

where $\pi^* = \mathrm{Hom}(\pi_*;\mathbf{Q})$, s denotes a degree 1 suspension, and h^* denotes the dual of the Hurewicz homomorphism. Let $l : \mathscr{M}(F_f)(2k-2) \to s^{-1}H^*(f)$ denote the obvious composition. Lift the cohomology of the map

$$\mathscr{M}(F_f)(2k-2) \xrightarrow{l} s^{-1}H^*(f) \xrightarrow{i^*} s^{-1}H^*(X)$$

to get a map $d_f : \mathscr{M}(F_f)(2k-2) \to \mathscr{M}(X)$ of degree 1 and define

$$\mathscr{M}'(A) = (\mathscr{M}(X) \otimes \mathscr{M}(F_f)(2k-2), d_X \otimes d_f).$$

It is then easy to write down a dga map $\mathscr{M}'(A) \to \mathscr{A}(A)$ which is a weak equivalence up to dimension $\min\{i+k, 2k-1\}$; the first bound is needed to exclude mixed products in cohomology. Since $H^*(A;\mathbf{Q}) = 0, * \ge j$, the rational homotopy of A can then be determined from $\mathscr{M}'(A)$ in a purely formal way [Sul77]. □

Corollary 2.2. *If $f_* : \pi_*(A) \otimes \mathbf{Q} \to \pi_*(X) \otimes \mathbf{Q}$ is onto for $k < * \le k'$, then $\mathscr{M}'(A)$ above is a* minimal *model through dimension $k'-1$. If $f_* : H_*(A;\mathbf{Q}) \to H_*(X;\mathbf{Q})$ is onto for $k < * \le k'$, then the differential d_f may be chosen to be trivial through dimension $k'-1$.*

Proof. Under the assumptions of the corollary, the lower horizontal maps in the following diagram are trivial:

$$\begin{array}{ccc}
\mathcal{M}(F_f)(2k-2) & \xrightarrow{d_f} & \mathcal{M}(X) \\
\downarrow l & & \\
H^{*+1}(f) & \longrightarrow & H^{*+1}(X) \\
\downarrow h^* & & \downarrow h^* \\
\pi^{*+1}(f) & \longrightarrow & \pi^{*+1}(X). \quad \square
\end{array}$$

Corollary 2.3. *Let* $f : A \to X$ *be as in* (2.1). *If* X *is formal, so is* A.

Proof. The diagram

$$\begin{array}{ccc}
\mathcal{M}(X) & \longrightarrow & H^*(X) \\
\downarrow & & \downarrow f^* \\
\mathcal{M}(X) \otimes \mathcal{M}(F_f)) & & H^*(A)
\end{array}$$

with a rational homotopy equivalence on top can easily be extended to a weak equivalence up to dimension $j-1$ in the bottom line as in (2.1). An extension to dimensions $\geq j$ is formal as well. $\square$

Definition 2.4. Let X be an n-dimensional connected 1-connected manifold, $F = \coprod F_j$ a collection of disjoint 1-connected submanifolds, dim $F_j = m_j$, $m = \max\{m_j\}$. Define $cX = \max\{k | H^k(X; \mathbf{Q}) = 0\}$ and

$$cF = \begin{cases} \max\{k | H^k(F; \mathbf{Q}) = 0\}, & F \text{ connected}, \\ -1, & \text{else.} \end{cases}$$

Assumption E. All of the following inequalities are satisfied:

$$m \leq cX + cF, \quad m < 2 \cdot cX - 1, \quad 2 \cdot m \leq n + 2 \cdot cF - 1, \quad 2 \cdot m \leq n + cX.$$

Corollary 2.5. *Under Assumption* E, *the rational homotopy type of* $M = X \setminus F$ *has cohomological dimension* $d = n - \min\{cX, cF + 1\}$ *and is determined by that of* X *and by the map* $i^* : H^*(X, M) \to H^*(X)$.

Proof. Use the Thom isomorphism (with $t(\nu_i)$ denoting the Thom class of ν_i), Alexander duality, and excision to obtain:

$$\begin{aligned} H^*(X, M) &\cong \bigoplus_i H^*(D\nu_i, S\nu_i) \\ &\cong \bigoplus_i H^{*-n+m_i}(F_i) \cdot t(\nu_i) = 0, \qquad * \leq n - m - 1; \end{aligned}$$

$$H^*(M) \cong H_{n-*}(M, S\nu) \cong H_{n-*}(X, F) = 0, \ * \geq n - cF - 1 \text{ and } * \geq n - cX.$$

Use (2.1) with $k = n - m - 1$, $j = \min\{n - cF - 1, \ n - cX\}$. $\square$

Corollary 2.6. *If, in addition to Assumption* E, *the inclusion of* F *into* X *is trivial, i.e., it factors over a point in rational homotopy, then a minimal model of* $M = X \setminus F$ *is obtained from* $\mathcal{M}(X) \otimes \Lambda s^{-1} H^*(X, M)$ *with trivial differential on the second factor by killing cohomology in dimensions greater than or equal to* $n - \min\{cX, cF + 1\}$. $\square$

Corollary 2.7. *If, in addition to Assumption* E, $\dim X \leq 4 \cdot cX + 2$, *then both* X *and* $M = X \setminus F$ *are formal.*

Proof. Combine [Mil79] with (2.3). □

The perturbation of X into $M \simeq X \setminus F$ that we have obtained above can be modified on the "Borel space level" to obtain a rational space $M_{[e]}$. We use the cohomological constructions from section 1 as a guide just as in [LR85, cf. in particular Satz 2.9]. We construct a minimal dga $(\mathcal{M}(M_{[e]}), \delta)$ such that $\mathcal{M}(M_{[e]}) \cong \mathcal{M}(M) \otimes \mathcal{M}(BT_{(0)}) \cong \mathcal{M}(M) \otimes \mathbf{Q}[e]$ as a graded algebra, and such that the differential δ on $\mathcal{M}(M_{[e]})$ satisfies the requirements of Assumption B for a deformation. Furthermore, its cohomolgy is the same as that of the candidate $H^*_{[e]}(M)$ constructed in section 1.

Proposition 2.8. *Under Assumptions* C *and* E, *there is a rational space* $M_{[e]}$ *over* $BT_{(0)}$ *and a rational homotopy equivalence from* M *into the pullback of the diagram*

$$\begin{array}{ccc} & & * = ET_{(0)} \\ & & \downarrow \\ M_{[e]} & \to & BT_{(0)}. \end{array}$$

Furthermore, there is an isomorphism $\psi : H^*(M_{[e]}) \to H^*_{[e]}(M)$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} & & H^*(M_{[e]}) & & \\ & \overset{t}{\nearrow} & & \overset{p}{\searrow} & \\ H^{*+1}(M) & & \downarrow \psi & & H^*(M). \\ & \underset{t_M}{\searrow} & & \underset{p_M}{\nearrow} & \\ & & H^*_{[e]}(M) & & \end{array}$$

Proof. We want to obtain a diagram

$$\begin{array}{ccc} \mathcal{M}(X) & \leftarrow & \mathcal{M}(X_{[e]}) \\ \hat{i} \downarrow & & \downarrow \hat{i}_{[e]} \\ \mathcal{M}(M) & \leftarrow & \mathcal{M}(M_{[e]}) \end{array},$$

where $\mathcal{M}(M_{[e]}) \cong \mathcal{M}(M) \otimes \mathbf{Q}[e]$ as a graded algebra, and $\mathcal{M}(M) \cong (\mathcal{M}(X) \otimes \Lambda s^{-1} H^*(X, M), d_X \otimes 0)$ as a dga, up to the cohomological dimension d from Corollary 2.5. (From Corollary 2.6 we know that the differential on the second factor is trivial.)

We have to find a perturbed differential on $\mathcal{M}(M_{[e]})$: First, we define a new differential $\delta_2 = e \cdot \tau : s^{-1} H^*(X, M) \to \mathcal{M}(X_{[e]})$ such that the map τ^* in cohomology makes the following diagram commute:

$$\begin{array}{ccc} s^{-1} H^*(X, M) & \hookrightarrow & H^*(M) \\ \tau^* \downarrow & & \downarrow t_M \\ H^{*+1}(X_{[e]}) & \overrightarrow{\overrightarrow{i_{[e]}}} & H^{*+1}_{[e]}(M). \end{array}$$

Extend τ to $\Lambda s^{-1} H^*(X, M)$ as a derivation, and define $\delta = d_X \otimes \delta_2$ to be the new differential on $\mathcal{M}(M_{[e]}))$ up to dimension d. Above the cohomological dimension d, the derivation can be extended formally—killing cohomology—in the same way as in the proof of [LR85, Satz 2.9]. The resulting cohomology $H^*(M_{[e]}) = H^*(\mathcal{M}(M_{[e]})$ is a quotient of $H^*(X_{[e]})$ which is isomorphic to $H^*_{[e]}(M)$, since the new differential annihilates precisely the kernel of $i_{[e]} : H^*(X_{[e]}) \to H^*_{[e]}(M)$. The map $i_{[e]}$ factors over $H^*(M_{[e]})$ to yield the map ψ above. $\square$

Remark 2.9. (1) The proof is by induction on degrees and works only for minimal algebras with *decomposable* differentials. I do not know whether (2.8) is true without Assumption C.

(2) One can show as in [LR85], that the construction above ends up with a formal space $X_{[e]}$ when feeded with a formal space X.

Note for the sake of completeness:

Lemma 2.10. *Let $\nu \downarrow F$ denote a vector bundle over the space F with minimal model $\mathcal{M}(F)$.*

(1) *If ν^l is a real vector bundle with Euler class $e(\nu) \in H^k(F)$, a minimal model for the sphere bundle $S(\nu)$ is given by*

$$\mathcal{M}(S\nu) = (\mathcal{M}(F) \otimes \bigwedge(s), d \otimes d_1),$$

where $|s| = l - 1$ and $d_1(s)$ is a cocyle representing $e(\nu)$ in cohomology.

(2) *If ν^k is a complex vector bundle with total Chern class $C(\nu) \in H^{**}(F)[e]$ (cf. 1.11), a minimal model of the projective bundle $\mathbf{C}P\nu$ is given by*

$$\mathcal{M}(\mathbf{C}P\nu) = (\mathcal{M}(F) \otimes \bigwedge(e, s), d \otimes d_2),$$

where $|e| = 2$, $|s| = 2k - 1$, $d_2(e) = 0$, and $d_2(s)$ is a cocycle representing $C(\nu)$.

We finish this section by lifting the map $k_{[e]} : H^*_{[e]}(M) \to H^*(\mathbf{C}P\nu)$ from Corollary 1.10 to rational homotopy.

Proposition 2.11. *Under Assumptions* C–E, *there is a dga map $K_{[e]} : \mathcal{M}(M_{[e]}) \to \mathcal{M}(\mathbf{C}P\nu)$ inducing $k_{[e]}$ in cohomology and fitting into the diagram*

$$\begin{array}{ccc}
\mathcal{M}(X_{[e]}) & \xrightarrow{j_{[e]}} & \mathcal{M}(F_{[e]}) \\
\downarrow i_{[e]} & & \downarrow p_{[e]} \\
\mathcal{M}(M_{[e]}) & \xrightarrow{K_{[e]}} & \mathcal{M}(\mathbf{C}P\nu) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\mathcal{M}(M) & \xrightarrow{k} & \mathcal{M}(S\nu).
\end{array}$$

Proof. Since $i^*_{[e]}$ is supposed to be onto, we only have to extend the map $p_{[e]} \circ j_{[e]} : \mathcal{M}(X_{[e]}) \to \mathcal{M}(\mathbf{C}P\nu)$ to $s^{-1}H^*(f)$ to get a dga map defined on $\mathcal{M}(M_{[e]})$, i.e., it has to be defined on the generators of $s^{-1}H^*(f)(d)$ in a way that commutes with differentials. Above the cohomological dimension, it can always be extended to a dga map.

Let y denote a generator from $s^{-1}H^*(X, M)$, representing a cocycle in $\mathcal{M}(M)$. We want to define $K_{[e]}(y) = k(y) + e \cdot k'(y)$, where k' has to be defined as a map of degree 2 in such a way that $K_{[e]}$ commutes with differentials. Note, that the lower diagram of the proposition commutes by definition. It turns out that $K_{[e]}$ is a dga map if and only if $d_{[e]}k'(y) = K_{[e]}(\tau y) - \tau k(y)$, where $d_{[e]}$

denotes the differential on $\mathscr{M}(\mathbf{C}P\nu)$ and τ denotes the transgressions in that algebra, resp. in $\mathscr{M}(M_{[e]})$. The element on the right side is in fact a coboundary by (1.13). □

3. Excision and duality

For the future development in [Rau94], it is important to ensure that the rational homotopy type of the pair given by the map $K_{[e]}$ of Proposition 2.11 from $\mathbf{C}P\nu$ into M behaves like a manifold with boundary, i.e., satifies Alexander duality. Furthermore, it is preferable to have excision properties at hand as in the situation of an actual T-action. In this section we assume the situation of Propositions 2.8 and 2.11, in particular, there is a commutative diagram of (rational) spaces:

$$\begin{array}{ccccc} S\nu & \to & M & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}P\nu & \to & M_{[e]} & \to & X_{[e]} \\ & \searrow & & \nearrow & \\ & & F_{[e]} & & \end{array}$$

Lemma 3.1. *The bottom maps give rise to an isomorphism* $H^*(X_{[e]}, F_{[e]}) \to H^*(M_{[e]}, \mathbf{C}P\nu)$. *In particular,*

$$H^*(M_{[e]}, \mathbf{C}P\nu) \cong \begin{cases} 0, & * \geq n, \\ \mathbf{Q}, & * = n-1. \end{cases}$$

Proof. By Proposition 2.11, there is an algebraic Gysin sequence

$$\cdots H^*(M, S\nu) \to H^{*-1}(M_{[e]}, \mathbf{C}P\nu) \to H^{*+1}(M_{[e]}, \mathbf{C}P\nu) \to \cdots$$

which is connected to the Gysin sequence of (X, F) and $(X_{[e]}, F_{[e]})$ by a ladder of homomorphisms. Excision yields an isomorphism $H^*(X, F) \cong H^*(M, S\nu)$ at every third term. As in [LR85], p. 564, the lemma follows by induction. □

Lemma 3.2. *Evaluation at a fundamental class in* $H^*(M_{[e]}, \mathbf{C}P\nu)$ *yields an isomorphism*

$$H^*(M_{[e]}) \to H_{n-1-*}(M_{[e]}, \mathbf{C}P\nu) \cong H_{n-1-*}(X_{[e]}, F_{[e]}).$$

Proof. Algebraic Gysin sequence, Alexander duality on total spaces, and induction as in [LR85], p.568. □

Lemma 3.2 is in fact the justification for choosing $H_{n-1-*}(X_{[e]}, F_{[e]})$ as the "cohomology candidate" $H^*_{[e]}(M)$.

Lemma 3.3. *The map* $H^*(X_{[e]}, M_{[e]}) \to H^*(F_{[e]}, \mathbf{C}P\nu)$ *is an isomorphism.*

Proof. Form the obvious ladder between the two Gysin sequences, use excision $H^*(X, M) \cong H^*(F, S\nu)$ on every third term, and induction. □

As in [Rau92], Proposition 3.1.3, one can conclude easily that the long exact sequence of the pair $(X_{[e]}, M_{[e]})$ splits into short exact pieces

$$0 \to H^*(X_{[e]}, M_{[e]}) \to H^*(X_{[e]}) \to H^*(M_{[e]}) \to 0.$$

4. Example: Actions on highly-connected manifolds with isolated fixed points

In [LR85], we showed the existence of $T_{(0)}$-actions on every CW-complex with the rational homotopy type of a sufficiently connected manifold with vanishing Euler-characteristic and index (and satisfying an additional technical assumption concerning Pontryagin classes). In this section, we want to generalize these results to manifolds with nonnegative Euler-characteristic and vanishing index, cf. Proposition 4.5. In [Rau94], they will be applied to the construction of semifree actions of cyclic groups with isolated fixed points on such manifolds. We recall from [LR85]:

Definition 4.1. A simply-connected CW-complex X is called an $FC2$-space, if $X_{(0)}$ is formal and has cup-length at most 2, i.e., $\tilde{H}^*(X;\mathbf{Q}) \cup \tilde{H}^*(X;\mathbf{Q}) \cup \tilde{H}^*(X;\mathbf{Q}) = 0$.

Remark 4.2. (1) [LR85] If X is rationally homotopy equivalent to a sphere or to a connected sum of products of spheres, i.e.,

$$X \simeq_{\mathbf{Q}} \sum_j^{\sharp}(S^{i_j} \times S^{n-i_j}), \qquad 2 \le i_j \le n - i_j,$$

then X is an $FC2$-space.

(2) [LR85] If M^n is a 1-connected manifold (Poincaré duality space suffices) with $H_i(M;\mathbf{Q}) = 0$ for $i \le k$ with $3k+1 \ge n$, then M is an $FC2$-space.

(3) If M^n is a 1-connected manifold with $H_i(M;\mathbf{Q}) = 0$ for $i \le k$ with $3k+1 \ge n$ and $\operatorname{index}(M) = 0$, then M is rationally homotopy equivalent to a sphere or a connected sum of products of spheres as in (1) above.

In the following, we shall concentrate on 1-connected manifolds X^n with

$$\begin{aligned} &n \text{ even},\ \chi(X) > 0,\ \operatorname{index}(X) = 0; \\ &H_i(X;\mathbf{Q}) = 0 \text{ for } i \le k \text{ with } 3k+1 \ge n. \end{aligned} \tag{4.1}$$

The corresponding case with $\chi(X) = 0$, in particular with n odd, is already treated in [LR85].

Some ideas and notations from [Rau92] are relevant in the following: Let X^n denote a manifold as in (4.1), on which T acts with $X^T \neq \varnothing$. The $\mathbf{Q}[e]$-module $H^*(X_T;\mathbf{Q})$ contains a $\mathbf{Q}[e]$-torsion submodule $\operatorname{Tor}(H^*(X_T;\mathbf{Q}))$. In [Rau92] we introduced and investigated the subspaces $I(X) = p^*(\operatorname{Tor}(H^*(X_T;\mathbf{Q}))) \subseteq R(X) = p^*H^*(X_T;\mathbf{Q}) \subseteq H^*(X;\mathbf{Q})$. An analysis of the derivation $t: H^*(X) \to H^{*-1}(X_T)$ in the Gysin sequence as in [LR85, Rau92] yields a $\mathbf{Q}$-linear map $\tau: H^*(X)/R(X) \to I(X)[e]$ that, combined with evaluation ev_1 at $e=1$, yields an isomorphism $ev_1 \circ \tau: H^*(X)/R(X) \to I(X)$ of $\mathbf{Q}$-vector spaces (of odd negativ degree). Similar to [LR85, Chapter 2], this gives rise to an isomorphism of $\mathbf{Q}[e]$-vector spaces

$$\operatorname{Tor}(H^*(X_T)) \cong I(X)[e]/e \cdot \tau(H^*(X)/R(X)),$$

and hence,

$$H^*(X_T) \cong I(X)[e]/e \cdot \tau(H^*(X)/R(X)) \oplus (R(X)/I(X))[e], \tag{4.2}$$

where we consider $I(X)$ and $R(X)/I(X)$ as $\mathbf{Q}$-vector spaces generating $H^*(X_T)$ as a $\mathbf{Q}[e]$-module. We can also describe the trivial part of the cup-product structure on $H^*(X_T)$:

Lemma 4.3. $\operatorname{Tor}(H^*(X_T)) \cup \tilde{H}^*(X_T) = \{0\} \subseteq H^*(X_T)$ *for a* T*-manifold* X *satisfying* 4.1.

Proof. All elements in $I(X) \cdot \tilde{H}^*(X_T)$ are $\mathbf{Q}[e]$-torsion in dimensions $\geq \frac{2n+4}{3}$; since τ is trivial in this range of dimensions, such a torsion element has to be trivial on the nose. □

Now, we proceed to construct the cohomology of a possible Borel space for some T-action on X, denoted as $H^*_{[e]}(X)$: If X is a rational homology sphere, we define

$$H^*_{[e]}(X) = H^*(S(n/2 \cdot U \oplus \mathbf{R})_T) \to 2 \cdot H^*(BT),$$

where U denotes a 1-dimensional free complex T-representation, and the map is induced from the inclusion map of the two fixed points of the action induced on the sphere at the Borel space level.

If X is not a rational homology sphere, (4.2) yields a rational homotopy equivalence

$$(4.3) \quad X \simeq_{\mathbf{Q}} N = \sideset{}{^\sharp}\sum_{1 \leq j \leq k} (S^{i_j} \times S^{n-i_j}) \sharp \sideset{}{^\sharp}\sum_{1 \leq j \leq k} (S^{i'_j} \times S^{n-i'_j}) \sharp \sideset{}{^\sharp}\sum_{1 \leq j \leq l} (S^{i''_j} \times S^{n-i''_j}),$$

where $i_j < \frac{n}{2}$ and odd, $i'_j, i''_j \leq \frac{n}{2}$ and even. In this case, $\chi(X) = 2l + 2$.

We introduce an auxiliary space $S = \sum^\sharp_{1 \leq j \leq l} (S^{i''_j} \times S^{n-i''_j})$, on which T acts semifreely with $2l + 2$ isolated fixed points as follows: Start with a linear action with two isolated fixed points of the form $S(U \oplus \mathbf{R})$, U a free T-representation, on each of the spheres. This produces actions with four isolated fixed points with the same tangential representations on every component in the connected sum. Taking a connected sum by identifying and eliminating fixed points in pairs yields the desired action with $\chi(S) = 2l + 2$ isolated fixed points.

Inclusion of the fixed point set induces a map $i_S : (2l + 2) \cdot BT \to S_T$ on the Borel space level, and the corresponding cohomology homomorphism $i_S^* : H^*(S_T) \to (2l + 2) \cdot H^*(BT)$ becomes an isomorphism after inverting e by the Borel localization theorem.

The cohomology fundamental classes of the spheres in N (cf. 4.3) will be denoted $x_j, y_j, x'_j, y'_j, x''_j$, resp. y''_j. Then $x_j \cup x'_j = x'_j \cup y'_j = x''_j \cup y''_j = [N]$, the cohomology fundamental class, whereas all other products vanish. Define

$$I(X) = \langle x_j, x'_j \rangle_{\mathbf{Q}}, \qquad R(X) = I(X) \oplus \langle x''_j, y''_j \rangle_{\mathbf{Q}},$$

and

$$\tau : H^*(X)/R(X) \to I(X)[e]$$

by

$$\tau([y_j]) = x'_j \cdot e^{\frac{n-i_j-i'_j-1}{2}}, \qquad \tau([y'_j]) = x_j \cdot e^{\frac{n-i'_j-i_j-1}{2}}.$$

We define, in accordance with (4.2) as a $\mathbf{Q}[e]$-module,

$$H^*_{[e]}(X) = I(X)[e]/e \cdot (\tau(H^*(X)/R(X)))[e] \oplus \langle y_j, y'_j \rangle_{\mathbf{Q}}[e].$$

Remark that $\langle y_j, y_j'\rangle_{\mathbf{Q}}[e] \cong H^*(S_T)$ as a $\mathbf{Q}[e]$-module; we use this isomorphism together with Lemma 4.3 to provide $H^*_{[e]}(X)$ with a graded product structure. There is a projection homomorphism $P : H^*_{[e]}(X) \to H^*(S_T)$ with kernel $I(X)[e]/(e\cdot\tau(H^*(X)/R(X)))[e]$, which gets isomorphic after inverting e. Together with the map i_S^* above, we have constructed a graded ring homomorphism

$$i_S^* \circ P : H^*_{[e]}(X) \to H^*(S_T) \to (2l+2)\cdot H^*(BT), \tag{4.4}$$

which gets an isomorphism after inverting e.

Finally, to get back from cohomology algebras to deformations of rational homotopy types, we apply

Lemma 4.4. *Let $\mathscr{H}_i^*$, $1 \le i \le 3$, denote graded 1-connected $\mathbf{Q}$-algebras, $\mathscr{H}_2^*$ moreover a $\mathbf{Q}[e]$-algebra, $|e| = 2$, which fit into an exact sequence*

$$\cdots \to \mathscr{H}_2^{*-2} \xrightarrow{\cdot e} \mathscr{H}_2^* \xrightarrow{P} \mathscr{H}_3^* \xrightarrow{\tau} \mathscr{H}_2^{*-1} \xrightarrow{\cdot e} \mathscr{H}_2^{*+1} \to \cdots, \tag{4.5}$$

where P is a ring homomorphism and τ a derivation. Let furthermore

$$\psi : \mathscr{H}_2^* \to \mathscr{H}_1^* \otimes \mathbf{Q}[e]$$

denote a $\mathbf{Q}[e]$-algebra homomorphism preserving e, which becomes an isomorphism after inverting e.

Then there are (formal) rational spaces F, X, $X_{[e]}$ and maps $X \xrightarrow{p} X_{[e]}$, resp. $F \times BT_{(0)} \xrightarrow{j_{[e]}} X_{[e]}$, that induce P, resp. ψ, in cohomology. In particular, $j_{[e]}^* : \mathscr{M}(X_{[e]}) \to \mathscr{M}(F \times BT_{(0)})$ satisfies Assumption B.

Proof. Let $\varphi_i : (\mathscr{A}_i, d_i) \to \mathscr{H}_i^*$, $i = 1, 2$, denote (formal 1-connected) dga minimal models. $\mathscr{A}_2$ contains a (unique) cocycle e of degree 2 representing the element of the same name in $\mathscr{H}_2^*$. Hence, the cokernel of the map $\cdot e : \mathscr{A}_2 \to \mathscr{A}_2$ is a (1-connected) dga $(\mathscr{A}_3, d_3)$, and φ_2 induces a dga map $\varphi_3 : \mathscr{A}_3 \to \operatorname{coker}(\cdot e : \mathscr{H}_2^* \to \mathscr{H}_2^*) \overset{P}{\hookrightarrow} \mathscr{H}_3^*$. Induction on the "Gysin" sequences (4.5) and of the quotient map $\mathscr{A}_2 \to \mathscr{A}_3$, cf. (1.1), shows that φ_3 induces an isomorphism in cohomology.

In fact, the quotient map $\mathscr{A}_2 \to \mathscr{A}_3$ induces P. Realizing the latter by rational spaces and maps [GM81, BG76] yields the map $p : X \to X_{[e]}$. According to [GM81], the map ψ can be lifted to give a dga map $\Psi : \mathscr{A}_2 \to \mathscr{A}_1 \otimes \mathbf{Q}[e]$. A realization of this map gives us the map $j_{[e]} : F \times BT_{(0)} \to X_{[e]}$. □

The following final result does not yet give us T-actions on the manifolds considered here, but should be considered as a first approximation step: Let X be a manifold satisfying (4.1) with $\chi = \chi(X) > 0$. Let U_i, $1 \le i \le \chi$, denote disjoint open disk neighbourhoods of χ points in X, and define $U = \bigcup_1^\chi U_i$, and $M = X \setminus U$. Its boundary ∂M may be (nonequivariantly) identified with the space $S = \bigcup_1^\chi S_i$, where $S_i = S(V)$, $1 \le i \le \chi$, and V denotes a complex free T-representation of real dimension n.

Proposition 4.5. *There is a rational space $\tilde{M}$ rationally homotopy equivalent to M supporting a free $T_{(0)}$-action. Furthermore, there is a $T_{(0)}$-equivariant map $I : S_{(0)} \to \tilde{M}$ that, up to rational cohomology, corresponds to the inclusion $S = \partial M \subset M$.*

Proof. Lemma 4.4 is applied to the case $\mathscr{H}_1^* = (2l+2)\cdot H^*(BT)$, $\mathscr{H}_2^* = H^*_{[e]}(X)$, $\mathscr{H}_3^* = H^*(X)$, and $\Psi = i_S^* \circ P$ from 4.4. As a result, we obtain a map $j_{[e]} : \chi \cdot BT_{(0)} \to X_{[e]}$, which satisfies Assumption B. We want to apply Propositions 2.8 and 2.11, and have to check Assumptions A through E. The inclusion of χ points into X obviously satisfies Assumptions A, C, and E. Assumption D is certainly satisfied (cf. Remark 1.9) when $X_{[e]}$ is replaced by S_T, cf. (4.4). Since the images of $H^*(X_{[e]})$ and of $H^*(S_T)$ in $H^*(\chi \cdot BT)$ coincide, Assumption D is satisfied for X as well.

Hence, Propositions 2.8 and 2.11 yield maps

$$\chi \cdot \mathbf{C}P_{(0)}^{\frac{n}{2}-1} \to M_{[e]} \to BT_{(0)}.$$

The pullback along these maps of the classifying fibre bundle $ET_{(0)} \downarrow BT_{(0)}$ yields the required equivariant map of free $T_{(0)}$-spaces $S_{(0)} \to \tilde{M} \simeq_{\mathbf{Q}} M$. □

Remark 4.6. For manifolds satisfying (4.1), but with negative Euler characteristic, one can perform a similar construction with $F = \sum_j^{\sharp}(S^3 \times S^3)$. Modulo fundamental group problems, one might also choose F to be an orientable surface.

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